



DYNAMICS OF ENVIRONMENTAL POLLUTION THROUGH VEHICLES

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Abstract:

In today's time as air pollution is increasing day by day the use of non-polluted has to be increased in almost all nooks and corner of the countries. In this paper a mathematical model is developed to analyse environmental pollution through polluted and non-polluted vehicles. Basic reproduction number has been calculated which will the decide the behavior of the system. Stability analysis has been carried out at equilibrium points. Numerical simulation is done to analyse the result for various compartments.

Keywords: *Vehicles; Mathematical Model; Environmental Pollution; System of non-linear Ordinary Differential Equation; Simulation.*

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1. Introduction

Pollution is one of the biggest problem the world faces today [3]. It occurs when harmful materials enter into air, water or soil. Various kinds of pollution prevail in the environment such as air pollution, water pollution, noise pollution, soil pollution, radioactive pollution, etc. Air pollution makes it difficult for plants, animals and human to survive as the air becomes dirty due to emission of harm full gases, dust, smoke in the atmosphere. Excessive air pollution is due to the vehicles.

Vehicles can be classified in to different categories as road vehicles, highway vehicles, own vehicles. Heavy duty vehicles and passenger vehicles are major contributors for creating pollution as they emit nitrogen oxides, carbon monoxide, hydrocarbon, etc [3][4]. These vehicles pollute many cities in India like Delhi, Kolkata, Mumbai, Chennai [6] and many more but WHO said that Delhi is the most polluted city [7]. Even more, DNA observed there were only few days in the year 2017 which had good air quality. Supreme court ordered Delhi government to reduce air pollution by [2],

- i. Pollution check by transport department.
- ii. Public awareness.
- iii. Catalytic converters.
- iv. Mass rapid transport vehicles.
- v. Pushing out of old commercial vehicles.
- vi. Tightening of mass ambition standard for new vehicles.

In this paper, we are targeting own vehicles which are classified into polluted and non-polluted. Non-polluted vehicles are those which are purchased recently or has work for 3-4 years. After that usage of some vehicles has been defined as polluted vehicles because the efficiency of its engine and other motor capacity. Conventional motor vehicles have high impact on environment pollution. Vehicles which are using gasoline, diesel or CNG are main source of causing air pollution [5].

In this paper, a mathematical model is formulated for the analysis of environmental pollution through vehicles. The notations, mathematical model and basic reproduction number are given in section 2. Section 3 discusses local and global stability which are evaluated at equilibrium points of the model. Sensitivity analysis and numerical simulation are discussed in section 4 and 5 to support the analytical result.

2. Mathematical Model

Here, we formulate a mathematical model for vehicles which are polluted or non-polluted. The model is developed with the notations defined in Table 1.

Figure 1 shows the dynamics of environmental pollution through vehicles. The model is developed with three discrete compartments: the number of vehicles (V), the number of polluted vehicles (P_V), the number of non-polluted vehicles (N_V).

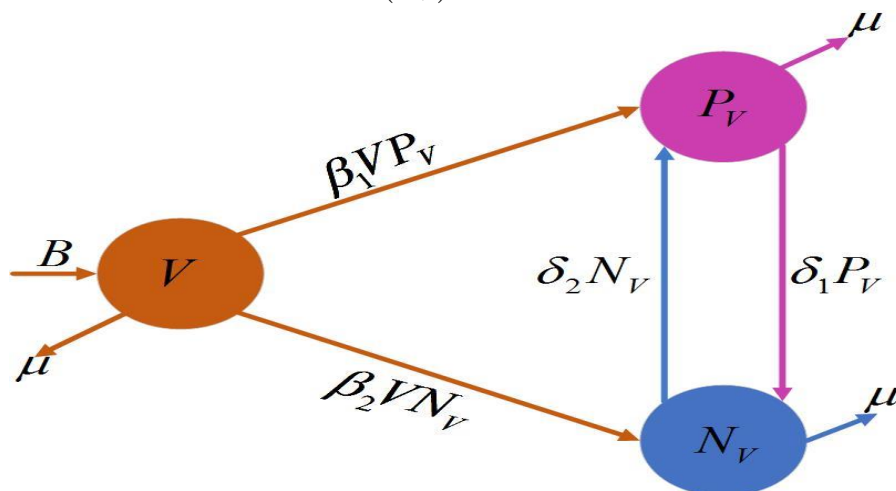


Figure 1: Transmission diagram of environmental pollution through vehicles

Table 1: Notation and its parametric values

| | | |
|------------|--|------|
| B | : New recruitment rate of vehicles | 0.12 |
| β_1 | : Rate of vehicles used for more than 5 years creating pollution | 0.1 |
| β_2 | : Rate of vehicles used for less than 5 years creating pollution | 0.25 |
| δ_1 | : Transfer rate of polluted to non-polluted vehicles | 0.7 |
| δ_2 | : Transfer rate of non-polluted to polluted vehicles | 0.06 |
| μ | : Rate of vehicles which are escaped from the observation | 0.3 |

Now, from figure 1 we have the following system of differential equation describing movement of vehicles from one compartment to the other

$$\begin{aligned} \frac{dV}{dt} &= B - \beta_1VP_v - \beta_2VN_v - \mu V \\ \frac{dP_v}{dt} &= \beta_2VP_v + \delta_2N_v - \delta_1P_v - \mu P_v \\ \frac{dN_v}{dt} &= \beta_2VP_v + \delta_1P_v - \delta_2N_v - \mu N_v \end{aligned} \tag{1}$$

with $V + P_v + N_v \leq N$ and $V > 0, P_v \geq 0, N_v \geq 0$.

Adding all the above equations, we have

$$\frac{d}{dt}(V + P_v + N_v) = B - \mu(V + P_v + N_v) \geq 0 \tag{2}$$

This gives, $\limsup_{t \rightarrow \infty}(V + P_v + N_v) \leq \frac{B}{\mu}$ (3)

Therefore, the feasible region for (1) is

$$A = \left\{ (V + P_v + N_v) : V + P_v + N_v \leq \frac{B}{\mu}, V > 0; P_v > 0, N_v > 0 \right\}. \tag{4}$$

Thus, the equilibrium point $E_0 = \left(\frac{B}{\mu}, 0, 0 \right)$.

Next, the basic reproduction number R_0 can be calculated using the next generation matrix [1].

Let $X = (P_v, N_v, V)$

Thus, $\frac{dX}{dt} = F(X) - V(X)$ (5)

$$F(X) = \begin{bmatrix} \beta_1VP_v \\ \beta_2VN_v \\ 0 \end{bmatrix}, \quad V(X) = \begin{bmatrix} \delta_1P_v + \mu P_v - \delta_2N_v \\ \delta_2N_v + \mu N_v - \delta_1P_v \\ -B + \beta_1VP_v + \beta_2VN_v + \mu V \end{bmatrix}$$

Now, the derivative of F and V of order 3×3 defined as

$$f = \left[\frac{\partial F_i(E_0)}{\partial X_j} \right], \quad v = \left[\frac{\partial V_i(E_0)}{\partial X_j} \right]$$

where

$$f = \begin{bmatrix} \beta_1V & 0 & \beta_1P_v \\ 0 & \beta_2V & \beta_2N_v \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} \delta_1 + \mu & -\delta_2 & 0 \\ -\delta_1 & \delta_2 + \mu & 0 \\ \beta_1V & \beta_2V & \beta_1P_v + \beta_2N_v + \mu \end{bmatrix}$$

Hence, the basic reproduction number R_0 is the spectral radius of matrix $f\nu^{-1}$

$$R_0 = \frac{B[\beta_1(\delta_2 + \mu) + \beta_2(\delta_1 + \mu)]}{\mu^2(\delta_1 + \delta_2 + \mu)} \tag{6}$$

On solving system (1), we get an endemic equilibrium point $E^* = (V^*, P_V^*, N_V^*)$

where, $V^* = A_1$, $P_V^* = \frac{-\delta_2(-A_1)\mu + B}{\mu A_1 \beta_1 - \delta_2 - \delta_1 - \mu}$,

$$N_V^* = \frac{A_1 \beta_1 (\beta_2 B - \delta_2 \mu - \mu^2) - \beta_2 B (\delta_1 + \mu) + \mu^2 (\mu + \delta_2 + \delta_1)}{\beta_1 \beta_2 \mu A_1 - \delta_2 - \delta_1 - \mu}$$

and $A_1 = \frac{\frac{1}{2} \beta_2 \delta_1 + \mu \beta_1 + \beta_2 \mu + \delta_2 \beta_1 \pm \sqrt{\beta_2^2 \delta_2^2 - 2 \beta_2 \delta_1 \mu + 2 \beta_2^2 \delta_1 \mu + 2 \beta_1 \beta_2 \delta_1 \delta_2 + \mu^2 \beta_1^2 - 2 \mu^2 \beta_1 \beta_2 - 2 \mu \beta_1^2 \delta_2 + \beta_2^2 \mu^2 - 2 \beta_1 \beta_2 \mu \delta_2 + \delta_2^2 \beta_1^2}}{\beta_1 \beta_2}$

3. Equilibrium

In this section, the local and global stability of environmental pollution through vehicles is discussed.

3.1. Local Stability

Theorem 1: If $\beta_1 \frac{B}{\mu} < \delta_1 + \mu$ and $\beta_2 \frac{B}{\mu} < \delta_2 + \mu$ then equilibrium point E_0 of the transmission of the model is locally asymptotically stable.

Proof: If all the eigenvalues of the Jacobian matrix of the system (1) have negative real part then the equilibrium point is locally stable. For this, at $E_0 = \left(\frac{B}{\mu}, 0, 0\right)$ the Jacobian of the system (1)

takes the form,

$$J = \begin{bmatrix} -\mu & \frac{-\beta_1 B}{\mu} & \frac{-\beta_2 B}{\mu} \\ 0 & \frac{\beta_1 B}{\mu} - \delta_1 - \mu & \delta_2 \\ 0 & \delta_1 & \frac{-\beta_2 B}{\mu} - \delta_2 - \mu \end{bmatrix}$$

$$trace(J) = -\mu + \frac{\beta_1 B}{\mu} - \delta_1 - \mu + \frac{\beta_2 B}{\mu} - \delta_2 - \mu$$

Now,

$$\text{trace}(J) = \min \left\{ \frac{\beta_1 B}{\mu(\delta_1 + \mu)}, \frac{\beta_2 B}{\mu(\delta_1 + \mu)} \right\} < 1 \tag{7}$$

Theorem 2: The unique positive equilibrium E^* is locally asymptotically stable with the condition that $a_{22} > 0, a_{33} > 0$ if and only if $\delta_1 + \mu > \beta_1 V^*$ and $\delta_2 + \mu > \beta_2 V^*$.

Proof: The Jacobian matrix of system (1) at endemic point E^* is as follows:

$$J^* = \begin{bmatrix} -a_{11} & -\beta_1 V^* & -\beta_2 V^* \\ \beta_1 P_V^* & -a_{22} & \delta_2 \\ \beta_2 N_V^* & \delta_1 & -a_{33} \end{bmatrix}$$

where, $a_{11} = \beta_1 P_V^* + \beta_2 N_V^* + \mu$, $a_{22} = -\beta_1 V^* + \delta_1 + \mu$, $a_{33} = -\beta_2 V^* + \delta_2 + \mu$

The characteristic polynomial of the above Jacobian matrix is

$$x^3 + A_{11}x^2 + A_{22}x + A_{33} = 0$$

where,

$$A_{11} = a_{11} + a_{22} + a_{33}$$

$$A_{22} = \beta_2^2 N_V V - \delta_1 \delta_2 + a_{33} a_{22} + a_{33} a_{11} + \beta_1^2 P_V V$$

$$A_{33} = \beta_2 N_V \beta_1 V \delta_2 + \delta_1 \beta_1 P_V \beta_2 V + \beta_2^2 N_V V a_{22} - \delta_1 \delta_2 a_{11} + a_{33} \beta_1^2 P_V V + a_{33} a_{22} a_{11}$$

Here, $A_{11} > 0, A_{22} > 0, A_{33} > 0$ and satisfy the condition $A_{11}A_{22} > A_{33}$ of Routh-Hurwitz criterion provided, $a_{22} > 0, a_{33} > 0$ (Routh E.J 1877).

3.2. Global Stability

In this section, we will discuss the global stability of the proposed model.

Theorem 3: The equilibrium point E_0 of the model is globally asymptotically stable.

Proof: If $\det(1 - fV^{-1}) > 0$ then the equilibrium is globally stable [8][9].

$$\text{Now, } \det(1 - fV^{-1}) = 1 - R_0 = 1 - 0.3597 = 0.6402 > 0 \tag{8}$$

Hence proved.

Theorem 4: The equilibrium point E^* of the model is globally asymptotically stable.

Proof: Consider, the Lyapunov function

$$L(t) = \frac{1}{2} \left[(V - V^*) + (P_V - P_V^*) + (N_V - N_V^*) \right]^2$$

then

$$\begin{aligned} L'(t) &= \left[(V - V^*) + (P_V - P_V^*) + (N_V - N_V^*) \right] \left[V' + P_V' + N_V' \right] \\ &= \left[(V - V^*) + (P_V - P_V^*) + (N_V - N_V^*) \right] \left[\mu V^* + \mu P_V^* + \mu N_V^* - \mu V - \mu P_V - \mu N_V \right] \\ &= -\mu \left[(V - V^*) + (P_V - P_V^*) + (N_V - N_V^*) \right]^2 \leq 0 \end{aligned}$$

Here, we used, $B = \mu V^* + \mu P_V^* + \mu N_V^*$.

Hence, E^* is globally asymptotically stable.

4. Sensitivity Analysis

In this section, the sensitivity analysis for all parameters is deliberated in Table 2.

The normalized sensitivity index of the parameters is calculated by using the following formula:

$$\gamma_\alpha^{R_0} = \frac{\partial R_0}{\partial \alpha} \cdot \frac{\alpha}{R_0}, \text{ where } \alpha \text{ denotes the model parameter.}$$

Table2.Sensitivity analysis

| Parameter | Value |
|------------|-------|
| B | + |
| β_1 | + |
| β_2 | + |
| δ_1 | - |
| δ_2 | - |
| μ | - |

New recruitment rate (B), the rate of vehicles used for more than 5 years creating pollution (β_1), the rate of vehicles used for less than 5 years creating pollution (β_2) have positive effect which means they pollutes surrounding other parameters have negative impact on model which advices individual to option for public transport and once polluted vehicles remains in the same state even if steps are taken to reduce pollution.

5. Numerical Simulation

In this section, we will study the numerical results of the model.

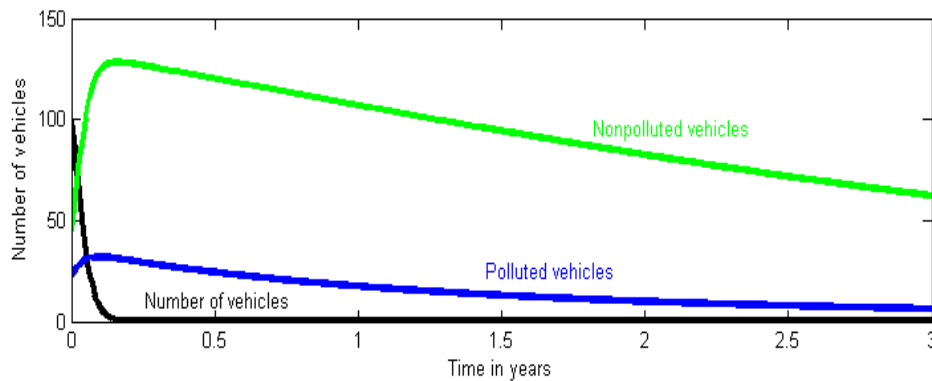


Figure 2: Movement of vehicles in compartments

Figure 2 interprets that the number of vehicles are decreasing due to which polluted and non-polluted vehicles are increasing. Non-polluted vehicles exist for more duration as compared to polluted vehicles and with the passage of time they both decreases and try to stabilize the pollution due to vehicles.

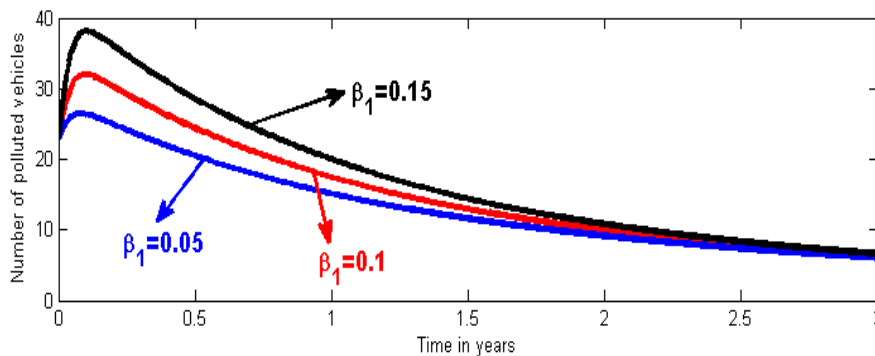


Figure 3: Effect of different values of β_1 on P_v

Figure 3 indicates that if the rate of vehicles used for more than five years creating pollution (β_1) is increased from 5% to 15% then number of polluted vehicles are decreasing approximately from 27 to 37. After 3 years, number of polluted vehicles have almost decreased and has become stable. so, the effect of vehicles used for more than 5 years creating pollution (β_1) play a vital role so pollution increases of such vehicles are used for longer time.

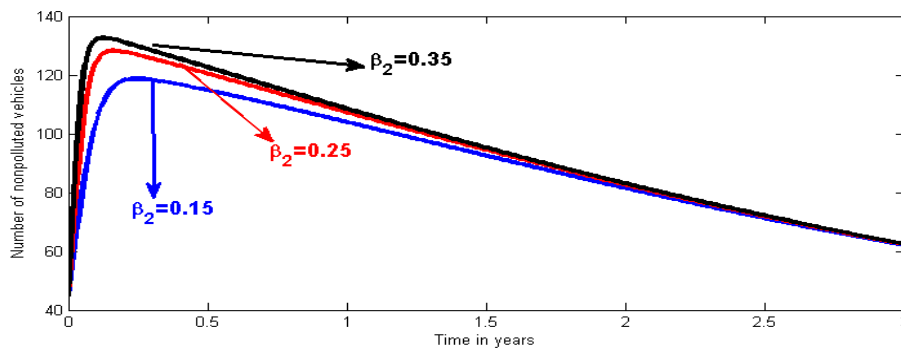


Figure 4: Effect of different values of β_2 on N_v

Now, from figure 4 it is observed that if one will increase the rate of vehicles used for less than 5 years creating pollution (β_2), the number of non-polluted vehicles decreases which is not advisable.

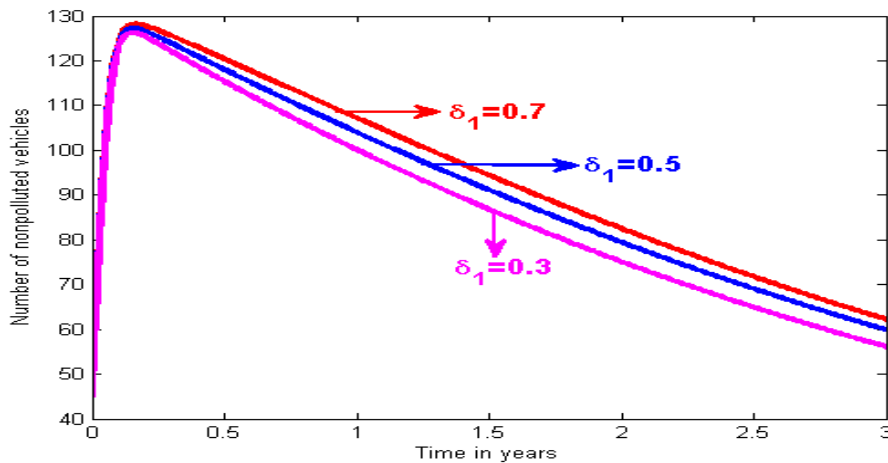


Figure 5: Effect of different values of δ_1 on N_V

Figure 5 shows that on decreasing the transfer rate of polluted to non-polluted vehicles (δ_1) the number of non-polluted vehicles also decreases which is harmful for the society.

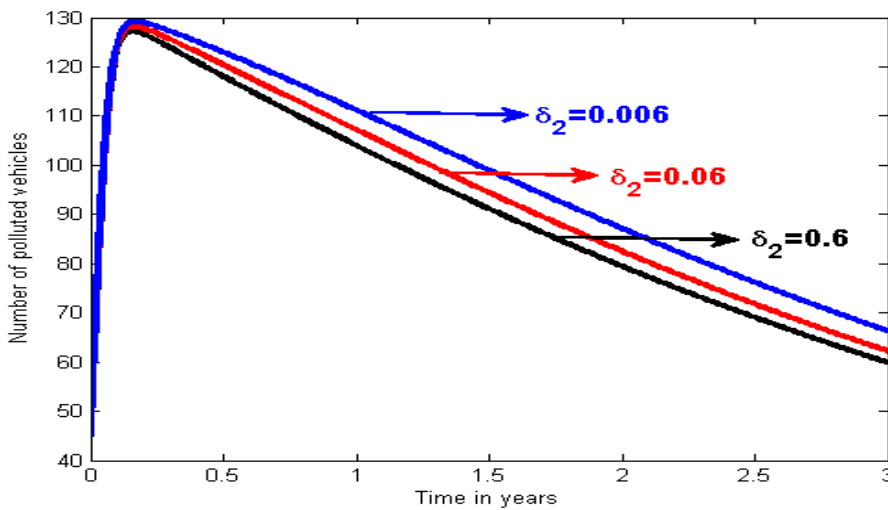


Figure 6: Effect of different values of δ_2 on P_V

Figure 6 specifies that if we increase the value from 0.06% to 0.6% initially. There is no increase or decrease in the number of polluted vehicles but with the passage of time they start to decrease. So, on increasing this rate the number of polluted vehicles decreases which is beneficial for the society.

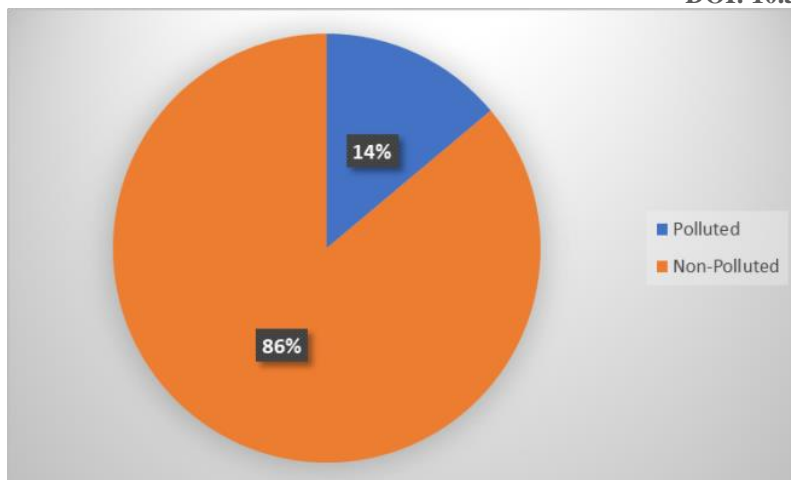


Figure 7: Percentage of polluted and non-polluted vehicles

Figure 7 suggests that though the number of non-polluted vehicles (86%) are more in comparison to polluted vehicles (14%). Every individual in the society should bring their step forward so as to bring maximum reduction in polluted vehicles to have a fresh and healthy surrounding.

6. Conclusions & Recommendations

A nonlinear mathematical model for environmental pollution through vehicles is developed. Local and global stability has been proved at equilibrium point E_0 . We conclude that increase in polluted vehicles cause more pollution which is harmful for environment and society. Instead of using polluted vehicles one should start using non-polluted vehicles or electric vehicles.

For that, some remedial steps can be taken such as:

- 1) Proper maintenance of car and truck emission control system.
- 2) Alternative fuels such as CNG and LPG.

Using parametric values given in the table 1. We computed that 35% of non-polluted vehicles exist in the society. The aim of the preparing model is to reduce environmental pollution through vehicles.

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