



## WHEEL RELATED ONE POINT UNION OF VERTEX PRIME GRAPHS AND INVARIANCE

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### Abstract:

*We investigate one point unions of  $W_4$ , and graphs obtained from  $W_4$  such as gear graph  $G_4$ , each cycle edge of  $W_4$  replaced with  $P_m$ , each pokes of  $W_4$  replaced with  $P_m$  for vertex prime labeling. All different non isomorphic structures of these graphs obtained by taking one point union graphs are shown to be vertex prime. This property of graphs is called as invariance under vertex prime labeling.*

**Keywords:** Labeling; Vertex Prime; Wheel; One Point Union.

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### 1. Introduction

The graphs we consider are finite, connected, and simple and un- directed. We refer F.Harary [4], Dynamic survey of graph labeling [3] for definitions and terminology. Deretsky, Lee, Mitchem Proposed a labeling called as vertex prime labeling of graph.[2].A function  $f : E(G) \rightarrow \{1,2,..|E|\}$  is such that for any vertex  $v$  the gcd of all labels on edges incident with  $v$  is 1. This is true to all vertices with degree at least 2.The graph that admits vertex prime labeling is called as vertex prime graph. They have shown that all forests, connected graphs,  $5C_{2m}$ , graph with exactly two components one of which is not odd cycle etc are vertex prime. One should refer A Dynamic survey of graph labeling by Joe Gallian to find further work done in this type of labeling.

In this paper we discuss vertex prime labeling of one point union of  $k$  copies of graphs. If we change the vertex on  $G$  at which one point union is taken, we may will get non isomorphic structure. We have studied non isomorphic one point unions of  $W_4$  and graphs obtained from  $W_4$  such as gear graph  $G_4$ , each cycle edge of  $W_4$  replaced with  $P_m$ , each pokes of  $W_4$  replaced with  $P_m$ , each cycle edge replaced with  $P_n$  and each pokes replaced with  $P_m$  for vertex prime labeling and have shown that they are vertex prime graphs. This is invariance under vertex prime labeling.

### 2. Preliminaries

- 1) A wheel graph  $W_n$  is obtained by taking a cycle  $C_n$  and a new vertex  $w$  outside of  $C_n$ .  $w$  is joined to each vertex of  $C_n$  by an edge each. It has  $2n$  edges and  $n+1$  vertices.

- 2)  $G^{(k)}$  it is One point union of k copies of G is obtained by taking k copies of G and fusing a fixed vertex of each copy with same fixed vertex of other copies to create a single vertex common to all copies. If we change the vertex on graph G used to fuse on k copies we may will get a different structure of one point union. If G is a (p, q) graph then  $|V(G_{(k)})| = k(p-1)+1$  and  $|E(G)| = k.q$
- 3) Fusion of vertex. Let G be a (p,q) graph. let  $u \neq v$  be two vertices of G. We replace them with single vertex w and all edges incident with u and that with v are made incident with w. If a loop is formed is deleted. The new graph has p-1 vertices and at least q-1 edges.[6]
- 4) A gear graph  $G_n$  is obtained from  $W_n$  by adding a new vertex on each edge of cycle  $C_n$  of  $W_n$ . It has  $2n+1$  vertices and  $3n$  edges.

### 3. Theorems Proved

#### 3.1. Theorem One point union of k copies of $W_4$ i.e. $G = (W_4)^{(k)}$ is vertex prime

Proof: G has  $4k + 1$  vertices and  $8k$  edges. The  $i^{th}$  copy of  $W_4$  in  $(W_4)^{(k)}$  be given ordinary labeling as follows : w as hub, and cycle vertices  $w_{i,1}, w_{i,2}, w_{i,3}, w_{i,4}$ . It's edge set is given by  $E(G) = \{c_{i,j}=(w_i w_{i,j})\}$  and  $\{e_{i,j}=(w_{i,j} w_{i,j+1}), j= 1,2,3,4, j+1$  taken (modulo 4) $\}$ .

Define a function  $f: E(G) \rightarrow \{1, 2, \dots, 8k\}$  as follows.

$$f(e_{i,j}) = 8(i-1) + j \text{ for } i = 1, 2, \dots, k \text{ and } j = 1, 2, 3, 4.$$

$$f(c_{i,j}) = f(e_{i,4}) + j.$$

All the consecutive cycle edges of  $C_4$  has label as consecutive numbers in addition to one edge with label 1. Therefore the gcd of labels incident on any vertex on cycle is one. The vertex  $w_i$  has at least four edges with label as consecutive integer. Label of hub  $w_i$  is 1. In the structure where one point union is taken at  $w_i$  as shown in figure the G has vertex prime label. Note that if we change the common vertex of one point union from hub to any vertex on cycle of  $W_4$  we will get another structure of  $G^{(k)}$ . These are the only two non-isomorphic structures possible on  $G^{(k)}$ . In both cases the same labeling function will work as vertex prime label function and the resultant graph is be vertex prime.

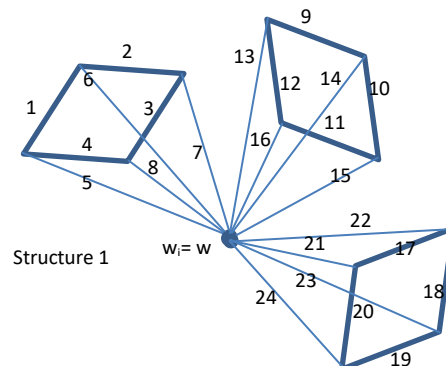
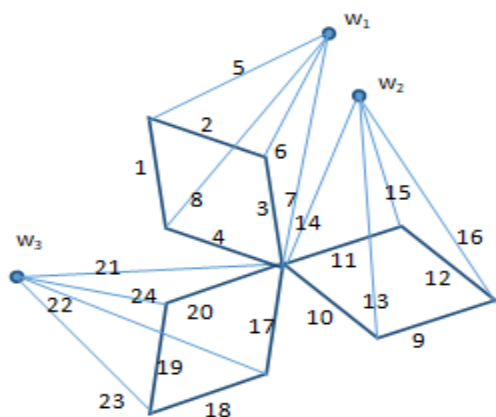


Fig 4.1:  $W_4^{(3)}$  with hub as common point: edge labels are shown.



Structure 2

Fig 4.2:  $W_4^{(3)}$  with hub as common vertex: edge labels are shown.

Thus there are two non-isomorphic structures possible on G and both are vertex prime. Thus G is invariant under vertex prime labeling.

### 3.2. Theorem One Point Union of $G_4$ I.E. $G = (G_4)^{(K)}$ is Vertex Prime

Proof:  $G_4$  has 9 vertices and 12 edges.  $G = (G_4)^{(K)}$  has  $8k+1$  vertices and  $12k$  edges. The  $i^{th}$  copy of  $G_4$  in  $(G_4)^{(k)}$  be given ordinary labeling as: was hub, the vertices on cycle  $C_8$  be  $w_{i,1}, w_{i,2}, \dots, w_{i,7}, w_{i,8}$ . It's edge set is given by  $E(G) = \{ c_{i,j}=(ww_{i,j})$  where  $j= 1,3,5,7$  and  $e_{i,j}=(w_{i,j}w_{i,j+1}), j = 1,2 \dots 8; j+1$  taken (modulo 8),for  $i = 1, 2, \dots, k$ .

Define a function  $f:E(G) \rightarrow \{1,2,\dots,12k\}$  as follows.

$$f(e_{i,j})=12(i-1)+j \text{ for } i = 1,2, \dots k \text{ and } j = 1,2,3,4.$$

$$f(c_{i,j})= f(e_{i,8})+j.$$

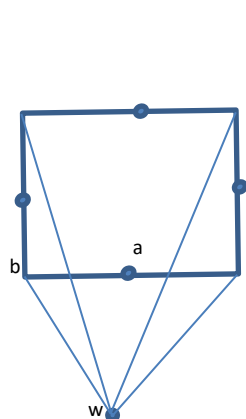


Fig. 4.3 Gear graph  $G_4$

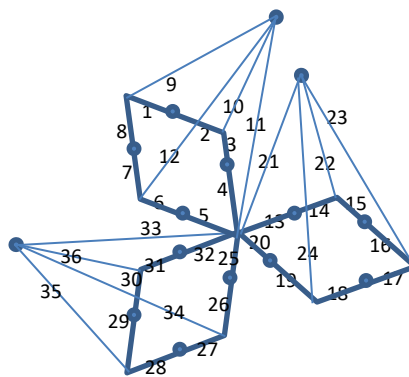


Fig 4.4:  $G_4^{(3)}$  with 'b' as common vertex: edge labels are shown.

On gear graph  $G_4$  we can take non-isomorphic one point union at points 'a' or 'b' or 'c' only. Thus there are only three non-isomorphic structures possible on G. For all these structures the labeling function f as above gives vertex prime labeling. All structures are vertex prime graphs.

**3.3. Theorem Let  $G'_4$  be a graph obtained from  $W_4$  by replacing each cycle edge (these edges are other than pokes) by a path  $P_m$ . Then  $G'_4$  and one point union of  $k$  copies of  $G'_4$  i.e.  $G = (G'_4)^{(K)}$  are vertex prime.**

Proof:  $G'_4$  has  $4m-3$  vertices and  $4m$  edges.  $G$  has  $(4m-5)k+1$  vertices and  $4mk$  edges. The  $i^{th}$  copy of  $G'_4$  in  $(G'_4)^{(k)}$  be given ordinary labeling as  $w_i$  as hub, the vertices on cycle  $C_{4m-4}$  be  $w_{i,1}, w_{i,2}, \dots, w_{i,m-1}, w_{i,m}, w_{i,m+1}, \dots, w_{i,4m-4}$ . Its edge set is given by  $E(G) = \{e_{i,j} = (w_i, w_{i,j}) \text{ where } j = 1, m, 2m-1, 3m-2; \text{ for } i = 1, 2, \dots, k\} \cup \{e_{i,j} = (w_{i,j}, w_{i,j+1}), j = 1, 2, \dots, 4m-4; j+1 \text{ taken (modulo } 4m-4)\}$ ; for  $i = 1, 2, \dots, k$ .

Define a function  $f: E(G) \rightarrow \{1, 2, \dots, 4mk\}$  as follows.  
 $f(e_{i,j}) = 4m(i-1) + j$  for  $i = 1, 2, \dots, k$  and  $j = 1, 2, \dots, 4m-4$ .  
 $f(c_{i,j}) = f(e_{i,4m-4}) + j; j = 1, 2, 3, 4$

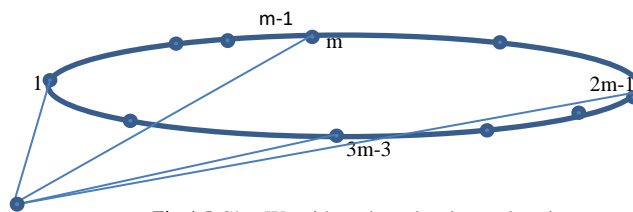


Fig 4.5  $G'_4 = W_4$  with each cycle edge replaced

We can take one point union at hub or at any of degree - 3 vertices on cycle (there are four such vertices and all four will produce isomorphic structure.) or at any degree two vertex on  $C_n$ . There are  $4m-8$  degree -2 vertices of which  $m-2$  vertices between  $v_1$  to  $v_m$  will produce non-isomorphic structures. Thus there can be  $m$  non-isomorphic structures possible on one point union of  $k$  copies of  $G'_4$ . The above function  $f$  will work for all structures and produce vertex prime labeling. Thus the graph is vertex prime.

**3.4. Theorem Let  $G'_4$  be a graph obtained from  $W_4$  by replacing each pokes edge (these edges are incident with hub) by a path  $P_m$ . Then  $G'_4$  and one point union of  $k$  copies of  $G'_4$  i.e.  $G = (G'_4)^{(K)}$  are vertex prime**

Proof:  $G'$  has  $4m-3$  vertices and has  $4m$  edges.  $G = (G'_4)^{(k)}$  has  $4km$  edges and  $4km-4k+1$  vertices. The  $i^{th}$  copy of  $G$  be given ordinary labeling as  $\{(v_{i1}e_{i1}v_{i2}e_{i2}v_{i3}e_{i3}v_{i4}e_{i4}) \text{ on cycle } C_4\}$  and on four paths for  $j = 1..4$  be given by  $(w_i v_{ij}) = \{w_i c_{i1}^j u_{i1} c_{i2}^j u_{i2} \dots u_{i(m-2)} c_{i(m-1)}^j v_{ij}\}$ .

Define a function  $f: E(G) \rightarrow \{1, 2, \dots, 4mk\}$  as follows.  
 $f(e_{i,j}) = 4m(i-1) + j$  for  $i = 1, 2, \dots, k$  and  $j = 1, 2, \dots, 4$ .  
 $f(c_{i,t}^j) = f(e_{i,4}) + (j-1)(m-1) + t; t = 1, 2, 3, \dots, (m-1), j = 1, 2, 3, 4; i = 1, 2, \dots, k$

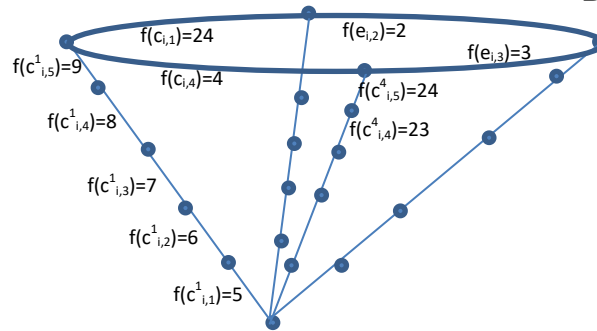


Fig 4.6  $G^*_4 = W_4$  with each pokes replaced with  $P_6$  with labels

**3.5. Theorem** Let  $G''_4$  be a graph obtained from  $W_4$  by replacing each cycle edge ( these edges are other than pokes) by a path  $P_m$  and each pokes is replaced by a path  $p_n$ . Then  $G''_4$  and one point union of  $k$  copies of  $G''_4$  i.e.  $G = (G''_4)^{(K)}$  with all structures are vertex prime

Proof: In  $G$  we define ordinary labeling of  $i^{th}$  copy of  $G''$  as on cycle of  $i^{th}$  copy we have  $\{v_{i1}e_{i1}v_{i2}e_{i2}..v_{i(n-1)}e_{i(n-1)}v_{im}\}$ . The four pokes are  $C^j = \{(w_i v^j_i), j = 1, 2, 3, 4\}$ . These are given by  $C^j = \{(w_i, c^j_1, u^j_{i1}, c^j_2, .., c^j_{i(n-1)}, u^j_{i(n-1)}, c^j_n, u^j_{in} / j = 1, 2, 3, 4; i = 1, 2, .., k\}$ . Note that  $u^j_{in} = v^j_i$  for  $j = 1, 2, 3, 4$  and  $i = 1, 2, .., k$ .  $G''$  has  $p = 4m + 4n - 7$  vertices and  $q = 4m + 4n - 8$  edges.  $G$  has  $kp - k + 1$  vertices and  $qk$  edges.

Define a function  $f: E(G) \rightarrow \{1, 2, \dots, 4mk\}$  as follows.  
 $f(e_{i,j}) = q(i-1) + j$  for  $i = 1, 2, \dots, k$  and  $j = 1, 2, \dots, q$ .  
 $f(c^j_{i,t}) = f(e_{i,q}) + (j-1)(n-1) + t; t = 1, 2, 3, \dots, (n-1), j = 1, 2, 3, 4; i = 1, 2, \dots, k$ .

Different pair wise non isomorphic structures can be obtained are  $n+m-2$ . These are obtained by choosing a common point as a point on poke  $P_n$  and  $m-2$  points between 3-degree consecutive vertices on cycle  $C_{4m-4}$ . The above function  $f$  will serve as vertex prime labeling for all these structures. Thus the graph  $G''$  and  $G = (G''_4)^{(K)}$  with all  $n+m-2$  non isomorphic structures of one point union of  $k$  copies of  $G''$  are vertex prime. That establishes the invariance under vertex prime labeling of  $G = (G''_4)^{(K)}$ .

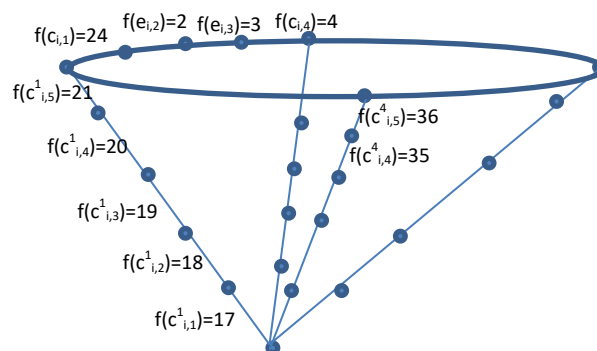


Fig 4.7  $G''_4 = W_4$  with each cycle edge replaced with  $p_5$  and pokes replaced with  $P_6$ .

#### 4. Conclusions

We have considered  $W_4$  and all of its related graphs such as gear graph  $G_4$  and each cycle edge of  $W_4$  replaced with  $P_m$ , each pokes of  $W_4$  replaced with  $P_m$  are shown to be vertex prime graphs. All different non isomorphic structures of these graphs obtained by taking one point union graphs are shown to be vertex prime. This property of graphs is called as invariance under vertex prime labeling. It is necessary to investigate these graphs further.

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