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MATHEMATICAL APPROACH ON HOUSEHOLD WASTE CAUSING ENVIRONMENTAL POLLUTANTS DUE TO LANDFILL AND TREATMENTS

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Abstract:
Household solid waste is the solid waste mixture of garbage and rubbish which comes during the use of various products in daily life. It also called as domestic waste or residential waste. It may fall into two categories either hazardous or non-hazardous which are stored and forsaken directly to the landfills. This is how household solid waste plays vital role in spreading environmental pollutants. For reduction of the pollution, treatment plant is constructed for hazardous solid waste and compost plant is organized for non-hazardous solid waste. In this paper, we have developed a system of non-linear differential equations to analyse the household solid waste storage. In order of preventive measures, five various controls are given to the different compartments. The basic reproduction number and the stability are derived to check the endurance of the model. The numerical simulation is also done using validated data.

Keywords: *Household Solid Waste; System of Non-Linear Differential Equations; Basic Reproduction Number; Local Stability; Global Stability; Environmental Pollutants.*

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1. Introduction

A solid waste is material which can be rejected if: Abandoned, Inherently waste-like, A discarded military munition or recycled in certain ways (https://www.epa.gov/hw/criteria-definition-solid-waste-and-solid-and-hazardous-waste-exclusions). Solid waste can be classified depending on their sources: (1) Household waste (2) Industrial waste (3) Biomedical waste. In the proposed paper, the prevention to environmental pollutants from household solid waste is found. Household solid waste is of bottles, cans, clothing, compost, disposable food packaging, food scraps, newspaper, magazines and yard trimmings which are released from private homes or apartments. In other words, household solid waste is refuse engendered only from households. Even though it does not contain huge number of germs but it entices flies, mosquitoes and rats, and allows them to breed which causes risk to human health. If household solid waste is kept even only for few hours, flies, mosquitoes and rats can be attracted. This may encourage the

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spread of diarrhoeal diseases as well dengue fever, yellow fever, bancroftian filariasis and bubonic plague. From all these wastes, certain types of household solid wastes are hazardous because they contain toxic elements. Household solid waste which can be categorised as hazardous solid waste covers batteries, shoe polish, paint tins, old medicines and medicine bottles. Rest of all waste are not consisting of toxic like paper, wood, fruits, vegetable peels, food and sludge considered as non-hazardous solid waste. Understanding how this waste is created is a useful factor of defining whether the waste is considered as hazardous or not. Everything leaves some waste after usage but it is required to take some special care when it is deposited. This is how proper waste management is an important fragment of the society. For large amount of waste, dumping in landfill is very common method. Some waste created from treatment plant and compost plant is also dumped in landfill. Treatment plant is the plant where hazardous solid waste is given treatment and compost plant is where non-hazardous solid waste is provided composting. These plants, landfill and storage of household solid waste make environment polluted. Hence, sprouting waste increase environmental pollutants. Therefore, waste disposal is becoming major problem of the society.

In 1999, Chongwoo and Iain examined an economic analysis of household waste management. Asa *et al.* studied life cycle assessment of energy from solid waste – part 2: landfilling compared to other treatment methods in 2005. Some researchers settled model on household waste. In 2005, forecasting municipal solid waste generation in a fast-growing urban region with system dynamics modeling is formed by Brian and Chang. Udaya and Achyuth in 2015 formulated mathematical modeling of household wastewater treatment by duckweed batch reactor. In 2008, Sara *et al.* developed mathematical modeling to predict residential solid waste generation. To revive nature some researchers have constructed mathematical models related to nature. In 2017, Nita *et al.* have established optimum control for spread of pollutants through forest resources and also expressed optimal control on depletion of green belt due to industries.

In this paper, a mathematical model is formulated in section 2. In section 3, local and global stability is defined. Optimal control is derived in section 4. Using given data, numerical simulation is considered in section 5.

2. Mathematical Modeling

We live in the society where the consumer market has grown quickly over the last few years. Everyone in the society uses the products being placed in cans, aluminum foils, plastic and other non-biodegradable items though they cause multitudinous damage to the environment. Also, some of these are hazardous wastes which are given treatment for betterment but it also causes harm to the environment. And non-hazardous items are provided the treatment as composting also cause the same problem. Therefore, we have measured eight different compartments for the model namely the tons of household solid waste (H_W), the tons of storage of household solid waste (S), the tons of hazardous household solid waste (H_Z), the tons of non-hazardous household solid waste (N_Z), the treatment plant (T), the compost plant (C), the landfill (L) and the environment pollutants (E_P). In this model, we have taken u_1 as the control rate of solid waste dumping, u_2 as the control rate on hazardous solid waste to give treatment, u_3 as the

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control rate on non-hazardous solid waste to provide composting, u_4 as the control rate for environment pollutants discharging from treatment plant and u_5 as the control rate for environment pollutants discharging from compost plant to diminish environmental pollutants.

The notation and parametric values of each parameters used in household solid waste model is given in the below table 1.

B	:	Recruitment rate of household solid waste	0.6
β	:	The rate at which household solid waste is stored	0.72
δ_1	:	The rate of hazardous household solid waste kept in storage	0.3
δ_2	:	The rate of non-hazardous household solid waste kept in storage	0.6
ε_1	:	The rate at which hazardous household solid waste goes to treatment plant	0.55
ε_2	:	The rate at which hazardous household solid waste goes to landfill	0.25
η_1	:	The rate at which non-hazardous household solid waste goes to compost plant	0.8
η_2	:	The rate at which non-hazardous household solid waste goes to landfill	0.4
γ_1	:	The rate of waste which comes to landfill from treatment plant	0.3
γ_2	:	The rate of waste which comes to landfill from compost plant	0.12
α_1	:	The rate of environment pollutants caused by storage	0.1
α_2	:	The rate of environment pollutants caused by treatment plant	0.3
α_3	:	The rate of environment pollutants caused by compost plant	0.25
α_4	:	The rate of environment pollutants caused by landfill	0.7
μ	:	Reduction rate of household solid waste from each compartment	0.4

Using above parameters and some necessary assumptions, the mathematical model is constructed. The transmission diagram of household solid waste model is as figure 1.

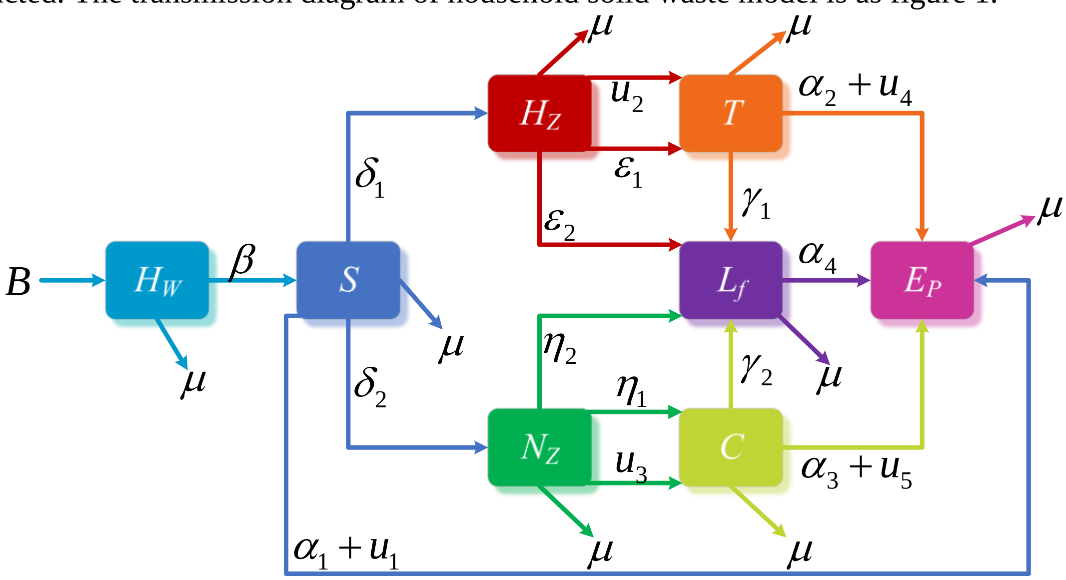

Figure 1: Transmission diagram for household solid waste model

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The system of non-linear differential equation of transmission of household solid waste is given below:

$$\begin{aligned}
\frac{dH_W}{dt} &= B - \beta H_W S - \mu H_W \\
\frac{dS}{dt} &= \beta H_W S - \delta_1 S - \delta_2 S - \alpha_1 S E_P - u_1 S - \mu S \\
\frac{dH_Z}{dt} &= \delta_1 S - \varepsilon_1 H_Z - \varepsilon_2 H_Z - u_2 H_Z - \mu H_Z \\
\frac{dN_Z}{dt} &= \delta_2 S - \eta_1 N_Z - \eta_2 N_Z - u_3 N_Z - \mu N_Z \\
\frac{dT}{dt} &= \varepsilon_1 H_Z - \alpha_2 T E_P - \gamma_1 T + u_2 H_Z - u_4 T - \mu T \\
\frac{dC}{dt} &= \eta_1 N_Z - \alpha_3 C E_P - \gamma_2 C + u_3 N_Z - u_5 C - \mu C \\
\frac{dL_f}{dt} &= \varepsilon_2 H_Z + \eta_2 N_Z + \gamma_1 T + \gamma_2 C - \alpha_4 L_f E_P - \mu L_f \\
\frac{dE_P}{dt} &= \alpha_1 S E_P + \alpha_2 T E_P + \alpha_3 C E_P + \alpha_4 L_f E_P + u_1 S + u_4 T + u_5 C - \mu E_P
\end{aligned} \tag{1}$$

where $H_W + S + N_Z + N_Z + T + C + L_f + E_P = N$.

Also, $H_W > 0; S, H_Z, N_Z, T, C, L_f, E_P \geq 0$.

Adding above differential equations of system (1), we get

$$\frac{d}{dt}\left(H_W + S + H_Z + N_Z + T + C + L_f + E_P\right) = B - \mu\left(H_W + S + H_Z + N_Z + T + C + L_f + E_P\right) \geq 0$$

Which implies that $\limsup_{t\to\infty}\left(H_W + S + H_Z + N_Z + T + C + L_f + E_P\right) \leq \frac{B}{\mu}$.

Therefore, the feasible region of the model is

$$\Lambda = \left\{\left(H_W, S, H_Z, N_Z, T, C, L_f, E_P\right) \in \mathbb{R}^8 : H_W + S + H_Z + N_Z + T + C + L_f + E_P \leq \frac{B}{\mu}\right\}.$$

Now, the equilibrium point of the household solid waste model is $E^*\left(H_W^*, S^*, H_Z^*, N_Z^*, T^*, C^*, L_f^*, E_P^*\right)$ where

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$$H_W^* = \frac{B}{\beta S^* + \mu}, S^* = S, H_Z^* = \frac{\delta_1 S^*}{\varepsilon_1 + \varepsilon_2 + \mu} N_Z^*, = \frac{\delta_2 S^*}{\eta_1 + \eta_2 + \mu},$$

$$T^* = \frac{\alpha_1 \varepsilon_1 \delta_1 S^* \left(\beta S^* + \mu\right)}{\left(\varepsilon_1 + \varepsilon_2 + \mu\right)\left(\alpha_2 \left(B\beta - \left(\delta_1 + \delta_2 + \mu\right)\left(\beta S^* + \mu\right)\right) + \alpha_1 \left(\gamma_1 + \mu\right)\left(\beta S^* + \mu\right)\right)},$$

$$C^* = \frac{\alpha_1 \eta_1 \delta_2 S^* \left(\beta S^* + \mu\right)}{\left(\eta_1 + \eta_2 + \mu\right)\left(\alpha_3 \left(B\beta - \left(\delta_1 + \delta_2 + \mu\right)\left(\beta S^* + \mu\right)\right) + \alpha_1 \left(\gamma_2 + \mu\right)\left(\beta S^* + \mu\right)\right)},$$

$$L_f^* = \frac{1}{\alpha_4 \left(B\beta - \left(\delta_1 + \delta_2 + \mu\right)\left(\beta S^* + \mu\right) + \mu \alpha_1 \left(\beta S^* + \mu\right)\right)} \left[\frac{\varepsilon_2 \delta_1 S^*}{\varepsilon_1 + \varepsilon_2 + \mu} + \frac{\eta_2 \delta_2 S^*}{\eta_1 + \eta_2 + \mu} \right.$$
$$+ \frac{\alpha_1 \gamma_1 \varepsilon_1 \delta_1 S^* \left(\beta S^* + \mu\right)}{\left(\varepsilon_1 + \varepsilon_2 + \mu\right)\left(\alpha_2 \left(B\beta - \left(\delta_1 + \delta_2 + \mu\right)\left(\beta S^* + \mu\right)\right) + \alpha_1 \left(\gamma_1 + \mu\right)\left(\beta S^* + \mu\right)\right)}$$
$$\left. + \frac{\alpha_1 \gamma_2 \eta_1 \delta_2 S^* \left(\beta S^* + \mu\right)}{\left(\eta_1 + \eta_2 + \mu\right)\left(\alpha_3 \left(B\beta - \left(\delta_1 + \delta_2 + \mu\right)\left(\beta S^* + \mu\right)\right) + \alpha_1 \left(\gamma_2 + \mu\right)\left(\beta S^* + \mu\right)\right)} \right],$$

$$E_P^* = \frac{B\beta - \left(\delta_1 + \delta_2 + \mu\right)\left(\beta S^* + \mu\right)}{\alpha_1 \left(\beta S^* + \mu\right)}$$

Next, we will compute the basic reproduction number noted as R_0 using the next generation matrix method.

Let us take $X' = \left(H_W, S, H_Z, N_Z, T, C, L_f, E_P\right)'$, where dash denotes derivative. So,

$$X' = \frac{dX}{dt} = f\left(X\right) - v\left(X\right)$$

where $f\left(X\right)$ is the rate of appearance of new individual in component and $v\left(X\right)$ is the rate of transfer of household solid waste. They are given by

$$f = \begin{bmatrix} \alpha_1 S E_P + \alpha_2 T E_P + \alpha_3 C E_P + \alpha_4 L_f E_P \\ 0 \\ \beta H_W S \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and}$$

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$$v = \begin{bmatrix} \mu E_P \\ -B + \beta H_W S + \mu H_W \\ \delta_1 S + \delta_2 S + \alpha_1 S E_P + \mu S \\ -\delta_1 S + \varepsilon_1 H_Z + \varepsilon_2 H_Z + \mu H_Z \\ -\delta_2 S + \eta_1 N_Z + \eta_2 N_Z + \mu N_Z \\ -\varepsilon_1 H_Z + \alpha_2 T E_P + \gamma_1 T + \mu T \\ -\eta_1 N_Z + \alpha_3 C E_P + \gamma_2 C + \mu C \\ -\varepsilon_2 H_Z - \eta_2 N_Z - \gamma_1 T - \gamma_2 C + \alpha_4 L_f E_P + \mu L_f \end{bmatrix}$$

Now, $Df\left(E^*\right) = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$ and $Dv\left(E^*\right) = \begin{bmatrix} V & 0 \\ J_1 & J_2 \end{bmatrix}$

where F and V are 8×8 matrices defined as

$$F = \left[\frac{\partial f_i\left(E^*\right)}{\partial X_j} \right] \text{ and } V = \left[\frac{\partial v_i\left(E^*\right)}{\partial X_j} \right].$$

Finding F and V, we get

$$F = \begin{bmatrix} \alpha_1 S^* + \alpha_2 T^* + \alpha_3 C^* + \alpha_4 L_f^* & 0 & \alpha_1 E_P^* & 0 & 0 & \alpha_2 E_P^* & \alpha_3 E_P^* & \alpha_4 E_P^* \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta S^* & \beta H_W^* & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{and}$$

$$V = \begin{bmatrix} \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta S^* + \mu & \beta H_W^* & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 S^* & 0 & \delta_1 + \delta_2 + \alpha_1 E_P^* + \mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\delta_1 & \varepsilon_1 + \varepsilon_2 + \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & -\delta_2 & 0 & \eta_1 + \eta_2 + \mu & 0 & 0 & 0 \\ \alpha_2 T^* & 0 & 0 & -\varepsilon_1 & 0 & \alpha_2 E_P^* + \gamma_1 + \mu & 0 & 0 \\ \alpha_3 C^* & 0 & 0 & 0 & -\eta_1 & 0 & \alpha_3 E_P^* + \gamma_2 + \mu & 0 \\ \alpha_4 L_f^* & 0 & 0 & -\varepsilon_2 & -\eta_2 & -\gamma_1 & -\gamma_2 & \alpha_4 E_P^* + \mu \end{bmatrix}$$

Here, above matrix V is non-singular matrix.

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Therefore, the expression of basic reproduction number R_0 is as following

$$R_0 = \text{spectral radius of matrix } FV^{-1}$$

$$\Rightarrow R_0 = \frac{\alpha_1 S^* + \alpha_2 T^* + \alpha_3 C^* + \alpha_4 L_f^*}{\mu} - \frac{\alpha_1^{\,2} E_P^* S^*}{\mu\left(\delta_1 + \delta_2 + \alpha_1 E_P^* + \mu\right)}$$
$$- \frac{\alpha_1 \alpha_2^{\,2} T^* E_P^{*2}\left(\varepsilon_1 + \varepsilon_2 + \mu\right) + \left(\left(\delta_1 + \delta_2 + \mu\right)\left(\varepsilon_1 + \varepsilon_2 + \mu\right)\alpha_2^{\,2} T^* + \alpha_1 \alpha_2 \varepsilon_1 \delta_1 S^*\right) E_P^*}{\mu\left(\varepsilon_1 + \varepsilon_2 + \mu\right)\left(\delta_1 + \delta_2 + \alpha_1 E_P^* + \mu\right)\left(\alpha_2 E_P^* + \gamma_1 + \mu\right)}$$
$$- \frac{\alpha_1 \alpha_3^{\,2} C^* E_P^{*2}\left(\eta_1 + \eta_2 + \mu\right) + \left(\left(\delta_1 + \delta_2 + \mu\right)\left(\eta_1 + \eta_2 + \mu\right)\alpha_3^{\,2} C^* + \alpha_1 \alpha_3 \eta_1 \delta_2 S^*\right) E_P^*}{\mu\left(\eta_1 + \eta_2 + \mu\right)\left(\delta_1 + \delta_2 + \alpha_1 E_P^* + \mu\right)\left(\alpha_3 E_P^* + \gamma_2 + \mu\right)}$$
$$- \frac{S^* f_1\left(E_P^*\right) + T^* f_2\left(E_P^*\right) + C^* f_3\left(E_P^*\right) + L_f^* f_4\left(E_P^*\right)}{\mu\left(\varepsilon_1 + \varepsilon_2 + \mu\right)\left(\eta_1 + \eta_2 + \mu\right)\left(\delta_1 + \delta_2 + \alpha_1 E_P^* + \mu\right)\left(\alpha_2 E_P^* + \gamma_1 + \mu\right)\left(\alpha_3 E_P^* + \gamma_2 + \mu\right)\left(\alpha_4 E_P^* + \mu\right)}$$

(2)

where

$$f_1\left(E_P^*\right) = \alpha_1 \alpha_4 \Big[\Big\{\alpha_2 \alpha_3\left(\delta_1 \varepsilon_2\left(\eta_1 + \eta_2 + \mu\right) + \delta_2 \eta_2\left(\varepsilon_1 + \varepsilon_2 + \mu\right)\right)\Big\} E_P^{*3} + \Big\{\alpha_3\left(\eta_2 \delta_2\left(\gamma_1 + \mu\right)\left(\varepsilon_1 + \varepsilon_2 + \mu\right) + \right.$$
$$\left(\eta_1 + \eta_2 + \mu\right)\delta_1\left(\left(\gamma_1 + \mu\right)\varepsilon_2 + \gamma_1 \varepsilon_1\right)\big) + \alpha_2\left(\left(\eta_1 + \eta_2 + \mu\right)\left(\delta_1 \varepsilon_2\left(\gamma_2 + \mu\right) + \eta_2 \delta_2 \varepsilon_1\right) + \delta_2\left(\varepsilon_2 + \mu\right)\right.$$
$$\left.\left(\eta_2\left(\gamma_2 + \mu\right) + \eta_1 \gamma_2\right)\right)\Big\} E_P^{*2} + \Big\{\left(\varepsilon_1 + \varepsilon_2 + \mu\right)\left(\gamma_1 + \mu\right)\left(\gamma_2 \eta_1 + \left(\gamma_2 + \mu\right)\eta_2\right)\delta_2 + \left(\eta_1 + \eta_2 + \mu\right)$$
$$\left(\gamma_2 + \mu\right)\left(\gamma_1 \varepsilon_1 + \left(\gamma_1 + \mu\right)\varepsilon_2\right)\delta_1\Big\} E_P^*\Big]$$

$$f_2\left(E_P^*\right) = \alpha_2 \alpha_4 \gamma_1\left(\varepsilon_1 + \varepsilon_2 + \mu\right)\left(\eta_1 + \eta_2 + \mu\right)\Big[\left\{\alpha_1 \alpha_3\right\} E_P^{*3} + \left\{\alpha_1\left(\gamma_2 + \mu\right) + \alpha_3\left(\delta_1 + \delta_2 + \mu\right)\right\} E_P^{*2} +$$
$$\left\{\left(\delta_1 + \delta_2 + \mu\right)\left(\gamma_2 + \mu\right)\right\} E_P^*\Big]$$

$$f_3\left(E_P^*\right) = \alpha_3 \alpha_4 \gamma_2\left(\varepsilon_1 + \varepsilon_2 + \mu\right)\left(\eta_1 + \eta_2 + \mu\right)\Big[\left\{\alpha_1 \alpha_2\right\} E_P^{*3} + \left\{\alpha_1\left(\gamma_1 + \mu\right) + \alpha_2\left(\delta_1 + \delta_2 + \mu\right)\right\} E_P^{*2} +$$
$$\left\{\left(\delta_1 + \delta_2 + \mu\right)\left(\gamma_1 + \mu\right)\right\} E_P^*\Big]$$

$$f_4\left(E_P^*\right) = \alpha_4^{\,2}\left(\varepsilon_1 + \varepsilon_2 + \mu\right)\left(\eta_1 + \eta_2 + \mu\right)\Big[\left\{\alpha_1 \alpha_2 \alpha_3\right\} E_P^{*4} + \Big\{\alpha_2\left(\varepsilon_1 + \varepsilon_2 + \mu\right)\left(\eta_1 + \eta_2 + \mu\right)\big(\alpha_1$$
$$\left(\gamma_1 + \mu\right) + \alpha_1 \alpha_3\left(\gamma_1 + \mu\right) + \alpha_3\left(\delta_1 + \delta_2 + \mu\right)\big)\Big\} E_P^{*3} + \Big\{\alpha_1\left(\gamma_1 + \mu\right)\left(\gamma_2 + \mu\right) + \alpha_2$$
$$\left(\delta_1 + \delta_2 + \mu\right)\left(\gamma_2 + \mu\right) + \alpha_3\left(\delta_1 + \delta_2 + \mu\right)\left(\gamma_1 + \mu\right)\Big\} E_P^{*2} + \Big\{\left(\delta_1 + \delta_2 + \mu\right)\left(\gamma_1 + \mu\right)$$
$$\left(\gamma_2 + \mu\right)\Big\} E_P^*$$

In the next section, equilibria of the household solid waste model will be conversed.

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3. Equilibrium

In the current section, the equilibrium of local stability and global stability is established for the household solid waste model.

3.1. Local Stability

Here, we have proposed the local stability of the household solid waste model at equilibrium point E^*.

Now, the Jacobian matrix J at the equilibrium point E^* is given by

$$J=\begin{bmatrix} -a_{11} & -\beta H_W^* & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta S^* & -a_{22} & 0 & 0 & 0 & 0 & 0 & -\alpha_1 S^* \\ 0 & \delta_1 & -a_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_2 & 0 & -a_{44} & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_1 & 0 & -a_{55} & 0 & 0 & -\alpha_2 T^* \\ 0 & 0 & 0 & \eta_1 & 0 & -a_{66} & 0 & -\alpha_3 C^* \\ 0 & 0 & \varepsilon_2 & \eta_2 & \gamma_1 & \gamma_2 & -a_{77} & -\alpha_4 L_f^* \\ 0 & \alpha_1 E_P^* & 0 & 0 & \alpha_2 E_P^* & \alpha_3 E_P^* & \alpha_4 E_P^* & -a_{88} \end{bmatrix}$$

where

$$a_{11}=\beta S^*+\mu, a_{22}=-\beta H_W^*+\delta_1+\delta_2+\alpha_1 E_P^*+\mu, a_{33}=\varepsilon_1+\varepsilon_2+\mu, a_{44}=\eta_1+\eta_2+\mu,$$
$$a_{55}=\alpha_2 E_P^*+\gamma_1+\mu, a_{66}=\alpha_3 E_P^*+\gamma_2+\mu, a_{77}=\alpha_4 E_P^*+\mu, a_{88}=-\alpha_1 S^*-\alpha_2 T^*-\alpha_3 C^*-\alpha_4 L_f^*+\mu$$

The characteristics polynomial of the Jacobian matrix J at E^* is of degree 8 with coefficients $A_{11}, A_{22}, A_{33}, A_{44}, A_{55}, A_{66}, A_{77}$ and A_{88} (file can be sent on request). Here, all the coefficients are positive and satisfy the condition of Routh-Hurwitz criterion (Routh E.J. 1877).

Theorem 1: The unique positive equilibrium point E^* is locally asymptotically stable with the condition that $A_{88}>0$ if and only if $\mu>\alpha_1 S+\alpha_2 T+\alpha_3 C+\alpha_4 L_f$.

3.2. Global Stability

Here, we have studied the global stability of the household solid waste model at equilibrium point E^*.

Consider the Lyapunov function

$$L(t)=\frac{1}{2}\Big[\left(H_W-H_W^*\right)+\left(S-S^*\right)+\left(H_Z-H_Z^*\right)+\left(N_Z-N_Z^*\right)+\left(T-T^*\right)+\left(C-C^*\right)+\left(L_f-L_f^*\right)$$
$$+\left(E_P-E_P^*\right)\Big]^2$$

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$$L'(t)=\left[\left(H_W-H_W^*\right)+\left(S-S^*\right)+\left(H_Z-H_Z^*\right)+\left(N_Z-N_Z^*\right)+\left(T-T^*\right)+\left(C-C^*\right)+\left(L_f-L_f^*\right)\right.$$
$$\left.+\left(E_P-E_P^*\right)\right]\left[H_W'+S'+H_Z'+N_Z'+T'+C'+L_f'+E_P'\right]$$
$$=\left[\left(H_W-H_W^*\right)+\left(S-S^*\right)+\left(H_Z-H_Z^*\right)+\left(N_Z-N_Z^*\right)+\left(T-T^*\right)+\left(C-C^*\right)+\left(L_f-L_f^*\right)\right.$$
$$\left.+\left(E_P-E_P^*\right)\right]\left[B-\mu H_W-\mu S-\mu H_Z-\mu N_Z-\mu T-\mu C-\mu L_f-\mu E_P\right]$$
$$=\left[\left(H_W-H_W^*\right)+\left(S-S^*\right)+\left(H_Z-H_Z^*\right)+\left(N_Z-N_Z^*\right)+\left(T-T^*\right)+\left(C-C^*\right)+\left(L_f-L_f^*\right)\right.$$
$$\left.+\left(E_P-E_P^*\right)\right]\left[\mu H_W^*+\mu S^*+\mu H_Z^*+\mu N_Z^*+\mu T^*+\mu C^*+\mu L_f^*+\mu E_P^*-\mu H_W-\mu S\right.$$
$$\left.-\mu H_Z-\mu N_Z-\mu T-\mu C-\mu L_f-\mu E_P\right]$$
$$=-\mu\left[\left(H_W-H_W^*\right)+\left(S-S^*\right)+\left(H_Z-H_Z^*\right)+\left(N_Z-N_Z^*\right)+\left(T-T^*\right)+\left(C-C^*\right)+\left(L_f-L_f^*\right)\right.$$
$$\left.+\left(E_P-E_P^*\right)\right]^2\leq 0$$

where we have used, $B=\mu H_W^*+\mu S^*+\mu H_Z^*+\mu N_Z^*+\mu T^*+\mu C^*+\mu L_f^*+\mu E_P^*$.

Theorem 2: The unique positive equilibrium point E^* is globally asymptotically stable.

4. Optimal Control

The objective of the household solid waste model is to minimize the environmental pollutants by maximizing treatments. To achieve the objective the controls are considered. For the mathematical model the objective function of household solid waste in the system along with the optimal controls is given as

$$J\left(u_i,\Omega\right)=\int_0^T\left(A_1H_W{}^2+A_2S^2+A_3H_Z{}^2+A_4N_Z{}^2+A_5T^2+A_6C^2+A_7L_f{}^2+A_8E_P{}^2+w_1u_1{}^2+w_2u_2{}^2\right.$$
$$\left.+w_3u_3{}^2+w_4u_4{}^2+w_5u_5{}^2\right)dt$$

(3)

where Ω signifies the set of all compartmental variables, $A_1,A_2,A_3,A_4,A_5,A_6,A_7,A_8$ means non-negative weight constants for $H_W,S,H_Z,N_Z,T,C,L_f,E_P$ compartments respectively and w_1,w_2,w_3,w_4,w_5 are weight constants for control variables u_1,u_2,u_3,u_4,u_5 respectively. Since, w_1,w_2,w_3,w_4,w_5 are constants for control rate of solid waste dumping, control rate on hazardous solid waste to give treatment, control rate on non-hazardous solid waste to provide composting, control rate for environment pollutants discharging from treatment plant and control rate for environment pollutants discharging from compost plant, respectively. u_1,u_4 and u_5 are the control variables to minimize the environmental pollutants comes from storage, treatment plant and compost plant. u_2 and u_3 are the control variables for maximizing the treatment to hazardous or non-hazardous solid waste from treatment plant or compost plant.

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To compute the values of control variables u_1, u_2, u_3, u_4 and u_5 from $t=0$ to $t=T$ such that

$$J\left(u_1(t), u_2(t), u_3(t), u_4(t), u_5(t)\right) = \min\left\{J\left(u_i^*, \Omega\right) \big/ \left(u_1, u_2, u_3, u_4, u_5\right) \in \Phi\right\}$$

where Φ represents a smooth function on interval [0,1]. The optimal control denoted by $u_i^*, i = 1,2,3,4,5$ are found by accumulating all the integrands of equation (3) using the lower bounds and upper bounds respectively with the results of Fleming *et al.* (2012).

Now, to minimize the cost function in equation (3) by creating Langragian function consisting of state equations and adjoint variables $A_i = \left(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8\right)$ using the Pontrygin's principle (1986) is defined as

$$\begin{aligned}
L\left(\Lambda, A_i\right) = {} & A_1 H_W{}^2 + A_2 S^2 + A_3 H_Z{}^2 + A_4 N_Z{}^2 + A_5 T^2 + A_6 C^2 + A_7 L_f{}^2 + A_8 E_P{}^2 \\
& + w_1 u_1{}^2 + w_2 u_2{}^2 + w_3 u_3{}^2 + w_4 u_4{}^2 + w_5 u_5{}^2 \\
& + \lambda_1\left(B - \beta H_W S - \mu H_W\right) \\
& + \lambda_2\left(\beta H_W S - \delta_1 S - \delta_2 S - \alpha_1 S E_P - u_1 S - \mu S\right) \\
& + \lambda_3\left(\delta_1 S - \varepsilon_1 H_Z - \varepsilon_2 H_Z - u_2 H_Z - \mu H_Z\right) \\
& + \lambda_4\left(\delta_2 S - \eta_1 N_Z - \eta_2 N_Z - u_3 N_Z - \mu N_Z\right) \\
& + \lambda_5\left(\varepsilon_1 H_Z - \alpha_2 T E_P - \gamma_1 T + u_2 H_Z - u_4 T - \mu T\right) \\
& + \lambda_6\left(\eta_1 N_Z - \alpha_3 C E_P - \gamma_2 C + u_3 N_Z - u_5 C - \mu C\right) \\
& + \lambda_7\left(\varepsilon_2 H_Z + \eta_2 N_Z + \gamma_1 T + \gamma_2 C - \alpha_4 L_f E_P - \mu L_f\right) \\
& + \lambda_8\left(\alpha_1 S E_P + \alpha_2 T E_P + \alpha_3 C E_P + \alpha_4 L_f E_P + u_1 S + u_4 T + u_5 C - \mu E_P\right)
\end{aligned} \tag{4}$$

The partial derivative of the Langragian function with respect to each variable of the compartment gives the adjoint equation variable $A_i = \left(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8\right)$ corresponding to the system which is as follows:

$$\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial L}{\partial H_W} = -\left(2A_1 H_W - \lambda_1 \beta S - \lambda_1 \mu + \lambda_2 \beta S\right) \\
&= -2A_1 H_W + \left(\lambda_1 - \lambda_2\right)\beta S + \lambda_1 \mu
\end{aligned} \tag{5}$$

$$\begin{aligned}
\dot{\lambda}_2 = -\frac{\partial L}{\partial S} = & -\left(2A_2 S - \lambda_1 \beta H_W + \lambda_2 \beta H_W - \lambda_2 \delta_1 - \lambda_2 \delta_2 - \lambda_2 \alpha_1 E_P - \lambda_2 u_1 - \lambda_2 \mu + \lambda_3 \delta_1 + \lambda_4 \delta_2 + \lambda_8 \alpha_1 E_P \right. \\
& \left. + \lambda_8 u_1\right) \\
= & -2A_2 S + \left(\lambda_1 - \lambda_2\right)\beta H_W + \left(\lambda_2 - \lambda_3\right)\delta_1 + \left(\lambda_2 - \lambda_4\right)\delta_2 + \left(\lambda_2 - \lambda_8\right)\alpha_1 E_P + \left(\lambda_2 - \lambda_8\right)u_1 + \lambda_2 \mu
\end{aligned}$$

(6)

$$\begin{aligned}
\dot{\lambda}_3 &= -\frac{\partial L}{\partial H_Z} = -\left(2A_3 H_Z - \lambda_3 \varepsilon_1 - \lambda_3 \varepsilon_2 - \lambda_3 u_2 - \lambda_3 \mu + \lambda_5 \varepsilon_1 + \lambda_5 u_2 + \lambda_7 \varepsilon_2\right) \\
&= -2A_3 H_Z + \left(\lambda_3 - \lambda_5\right)\varepsilon_1 + \left(\lambda_3 - \lambda_7\right)\varepsilon_2 + \left(\lambda_3 - \lambda_5\right)u_2 + \lambda_3 \mu
\end{aligned} \tag{7}$$

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$$\dot{\lambda}_4 = -\frac{\partial L}{\partial N_Z} = -\left(2A_4N_Z - \lambda_4\eta_1 - \lambda_4\eta_2 - \lambda_4u_3 - \lambda_4\mu + \lambda_6\eta_1 + \lambda_6u_3 + \lambda_7\eta_2\right) \\ = -2A_4N_Z + (\lambda_4 - \lambda_6)\eta_1 + (\lambda_4 - \lambda_7)\eta_2 + (\lambda_4 - \lambda_6)u_3 + \lambda_4\mu \qquad (8)$$

$$\dot{\lambda}_5 = -\frac{\partial L}{\partial T} = -\left(2A_5T - \lambda_5\alpha_2E_P - \lambda_5\gamma_1 - \lambda_5u_4 - \lambda_5\mu + \lambda_7\gamma_1 + \lambda_8\alpha_2E_P + \lambda_8u_4\right) \\ = -2A_5T + (\lambda_5 - \lambda_8)\alpha_2E_P + (\lambda_5 - \lambda_7)\gamma_1 + (\lambda_5 - \lambda_8)u_4 - \lambda_5\mu \qquad (9)$$

$$\dot{\lambda}_6 = -\frac{\partial L}{\partial C} = -\left(2A_6C - \lambda_6\alpha_3E_P - \lambda_6\gamma_2 - \lambda_6u_5 - \lambda_6\mu + \lambda_8\alpha_3E_P + \lambda_8u_5 + \lambda_7\gamma_2\right) \\ = -2A_6C + (\lambda_6 - \lambda_8)\alpha_3E_P + (\lambda_6 - \lambda_7)\gamma_2 + (\lambda_6 - \lambda_8)u_5 + \lambda_6\mu \qquad (10)$$

$$\dot{\lambda}_7 = -\frac{\partial L}{\partial L_f} = -\left(2A_7L_f - \lambda_7\alpha_4E_P - \lambda_7\mu + \lambda_8\alpha_4E_P\right) \\ = -2A_7L_f + (\lambda_7 - \lambda_8)\alpha_4E_P + \lambda_7\mu \qquad (11)$$

$$\dot{\lambda}_8 = -\frac{\partial L}{\partial E_P} = -\left(2A_8E_P - \lambda_2\alpha_1S - \lambda_5\alpha_2T - \lambda_6\alpha_3C - \lambda_7\alpha_4L_f + \lambda_8\alpha_1S + \lambda_8\alpha_2T + \lambda_8\alpha_3C + \lambda_8\alpha_4L_f - \lambda_8\mu\right) \\ = -2A_8E_P + (\lambda_2 - \lambda_8)\alpha_1S + (\lambda_5 - \lambda_8)\alpha_2T + (\lambda_6 - \lambda_8)\alpha_3C + (\lambda_7 - \lambda_8)\alpha_4L_f + \lambda_8\mu \qquad (12)$$

The necessary condition for Langragian function L to be optimal for controls is

$$\dot{u}_1 = -\frac{\partial L}{\partial u_1} = -2w_1u_1 + (\lambda_2 - \lambda_8)S \qquad (13)$$

$$\dot{u}_2 = -\frac{\partial L}{\partial u_2} = -2w_2u_2 + (\lambda_3 - \lambda_5)H_Z \qquad (14)$$

$$\dot{u}_3 = -\frac{\partial L}{\partial u_3} = -2w_3u_3 + (\lambda_4 - \lambda_6)N_Z \qquad (15)$$

$$\dot{u}_4 = -\frac{\partial L}{\partial u_4} = -2w_4u_4 + (\lambda_5 - \lambda_8)T \qquad (16)$$

$$\dot{u}_5 = -\frac{\partial L}{\partial u_5} = -2w_5u_5 + (\lambda_6 - \lambda_8)C \qquad (17)$$

Solving the above equations (13) - (17), we found the values of u_1, u_2, u_3, u_4 and u_5 as

$$u_1 = \frac{(\lambda_2 - \lambda_8)S}{2w_1}, u_2 = \frac{(\lambda_3 - \lambda_5)H_Z}{2w_2}, u_3 = \frac{(\lambda_4 - \lambda_6)N_Z}{2w_3}, u_4 = \frac{(\lambda_5 - \lambda_8)T}{2w_4} \text{ and } u_5 = \frac{(\lambda_6 - \lambda_8)C}{2w_5}$$

Thus, the required optimal control condition is calculated as

$$u_1^* = \max\left(a_1, \min\left(b_1, \frac{(\lambda_2 - \lambda_8)S}{2w_1}\right)\right)$$

$$u_2^* = \max\left(a_2, \min\left(b_2, \frac{(\lambda_3 - \lambda_5)H_Z}{2w_2}\right)\right)$$

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$$u_3^* = \max\left(a_3, \min\left(b_3, \frac{(\lambda_4 - \lambda_6) N_Z}{2w_3}\right)\right) \quad (18)$$

$$u_4^* = \max\left(a_4, \min\left(b_4, \frac{(\lambda_5 - \lambda_8) T}{2w_4}\right)\right)$$

$$u_5^* = \max\left(a_5, \min\left(b_5, \frac{(\lambda_6 - \lambda_8) C}{2w_5}\right)\right)$$

Next, to support the analytical result we have considered the optimal control numerically.

5. Numerical Simulation

In this section, we detected the numerical simulation using the parametric values mentioned in Table 1.

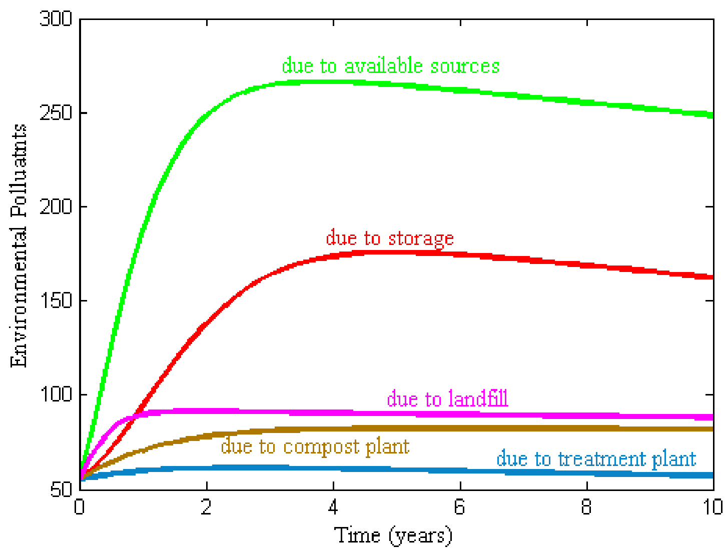

Figure 2: Effect of rate of garbage on environmental pollutants

Above figure 2 represents the effect of garbage related to different sources on environmental pollutants. In the proposed paper, there are four ways for creating environmental pollutants:

i. Due to the storage
ii. Due to the treatment plant
iii. Due to the compost plant
iv. Due to the landfill

Here, due to all available sources, environmental pollutants are growing extra in compared to spreading only through the storage, treatment plant, compost plant and landfill. Due to all sources environmental pollutants increase by 250. On the other hand, due to only storage, treatment plant, compost plant and landfill, environmental pollutants are increasing approximately by 162, 57, 81 and 88 respectively. This suggests that we should apply control on each source.

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Next four figures show the effect of control on – storage, treatment plant, compost plant, landfill – the compartments.

From figure 2, it is observed that treatment plant causes less pollution in comparison to other sources. So, one can apply less control on it. Therefore, after applying control, treatment plant rises with smaller rate which can be noticed from figure 3.

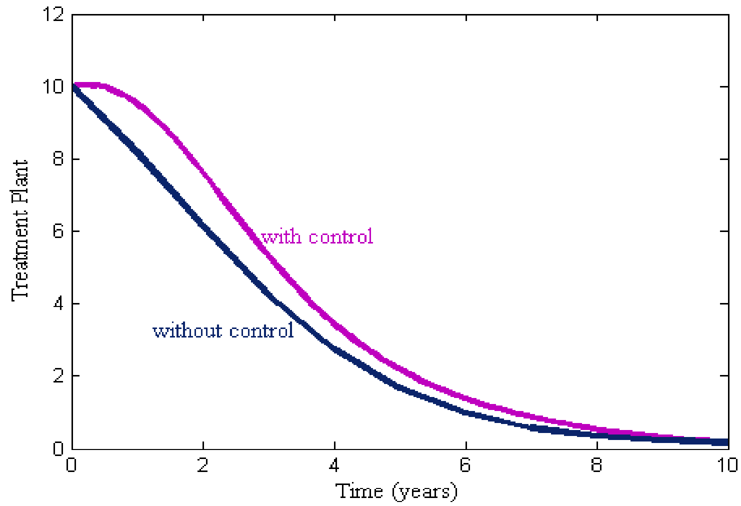

Figure 3: Effect of treatment plant with control and without control

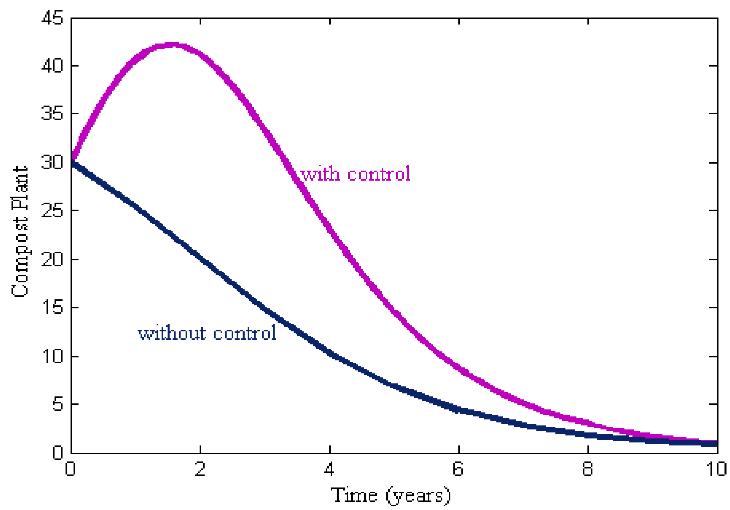

Figure 4: Effect of compost plant with control and without control

Figure 4 suggests the behaviour of compost plant before and after applying control. Before applying control, compost plant is falling rapidly. But when control is applied, it increases by 30 to 42 in just about 1.5 years which is helpful to revive the nature.

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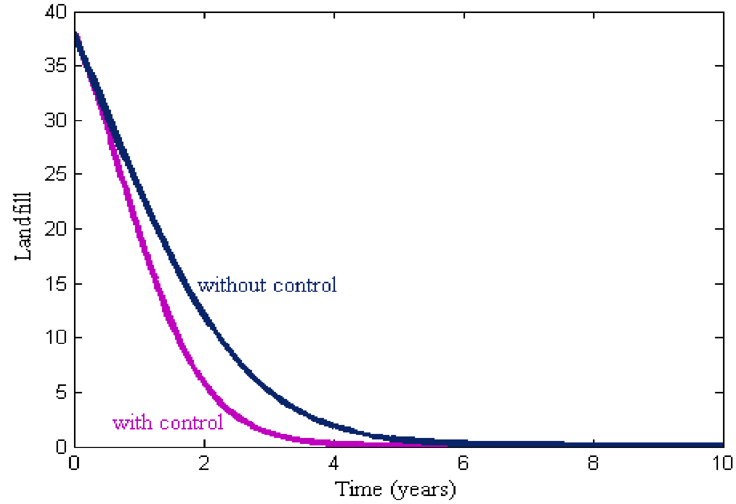

Figure 5: Effect of landfill with control and without control

As landfill also causes pollution, it is required to have control on it. Figure 5 advocates, how much landfill is decreased after putting control. After applying control, it decreases by approximately 65.25% in almost 2.2 years.

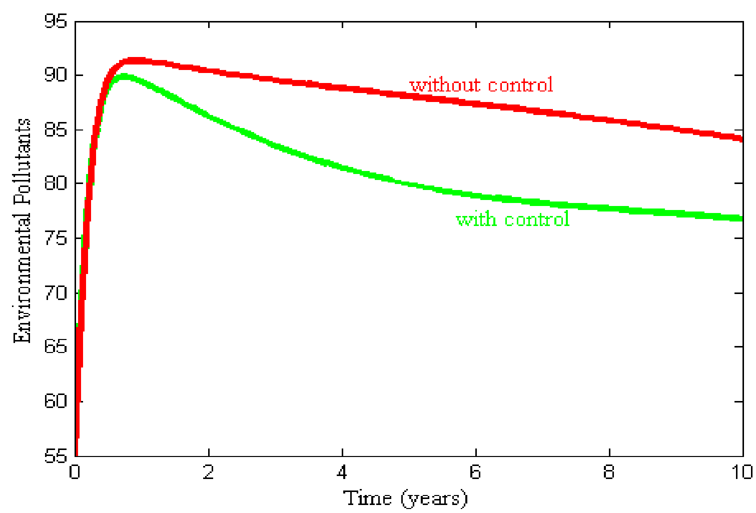

Figure 6: Effect of environment pollutants with control and without control

To control the environment pollutants, we have applied some controls. The effect of these controls on environment pollutants can be seen from figure 6. Initially environment pollutants are increasing and then after control is given they start decreasing. Here, environment pollutants are decreased by 84.15 to 76.76 unit in 10 years which indicates that if controls are exist then we can revive environment by 8.76%.

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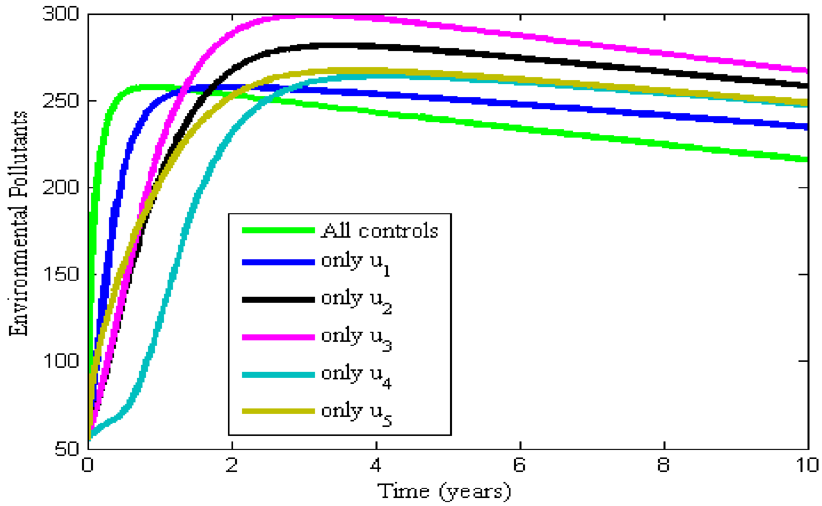

Figure 7: Effect of control variables

In this figure 7, we will see the effect of individual control on environmental pollutants. For this we have taken six cases:

i. All the controls are applied. i.e. $u_i \neq 0; i = 1,2,3,4,5.$

ii. Only u_1 – control rate of solid waste dumping – is applied. i.e. $u_1 \neq 0; u_i = 0; i = 2,3,4,5.$

iii. Only u_2 – control rate on hazardous solid waste to give treatment – is applied. i.e. $u_2 \neq 0; u_i = 0; i = 1,3,4,5.$

iv. Only u_3 – control rate on non-hazardous solid waste to provide composting – is applied. i.e. $u_3 \neq 0; u_i = 0; 1,2,4,5.$

v. Only u_4 – control rate for environment pollutants discharging from treatment plant – is applied. i.e. $u_4 \neq 0; u_i = 0; i = 1,2,3,5.$

vi. Only u_5 – control rate for environment pollutants discharging from plant – is applied. i.e. $u_5 \neq 0; u_i = 0; i = 1,2,3,4.$

From figure 7, we specify that when only u_3 control is applied then environmental pollutants are increasing a lot. Now, if when only u_2 control is applied then environmental pollutants are little less than u_3 and so on for only u_1, u_4 and u_5 are applied. But when all the controls are applied then environmental pollutants are reduced. It means environment is getting revive when all the controls are presenting their role. So, it is advisable to have all the controls for saving environment as well as human health.

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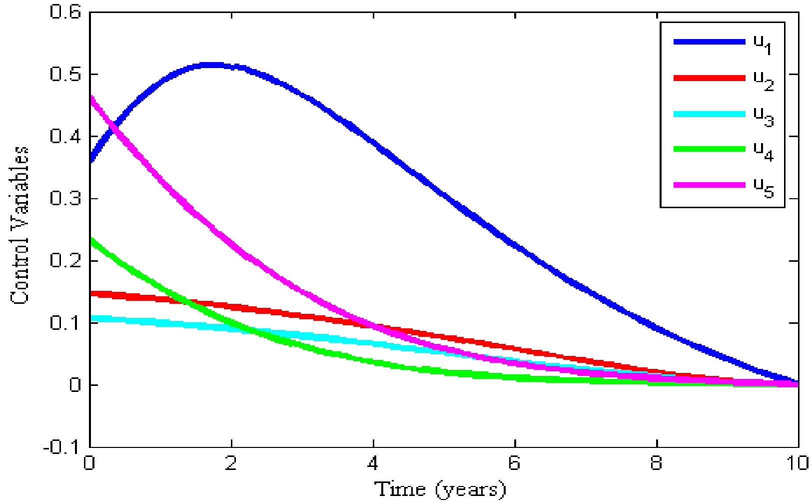

Figure 8: Control variables

As controls are good for saving the environment, figure 8 suggests that the highest control is required for u_1 on storage by 51% in 1.8 years and then can be decreased. For u_2 and u_3 corresponding 14% and 10% controls will be required in the beginning and can be decreased. Similarly, 23% and 46% control will be needed for u_4 and u_5 . For the betterment of the environment we should apply appropriate controls rates as mentioned above.

6. Conclusion

In this paper, a mathematical model of household solid waste is formulated with the aim of preventing environmental pollutants. A mathematical approach on household causing environmental pollutants due to landfill and treatments is to study how much landfill and treatments pollute the environment. We all know that proper waste disposal is useful to save the nature. We should dump our household waste in appropriate container. Also, we can shop for environmentally friendly natural product or else we can reduce our purchase of products that consists of non-reusable ingredients. We can consider the alternative methods to use products for some common household needs. Also, we can search online for simple recipes that can use to create by own.

Using the parametric values given in the table 1, the basic reproduction number is intended as 0.2734 which indicates that the controls are beneficial to save the environment. This advocates that proper planning of waste disposal is useful to make environment and human healthy.

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